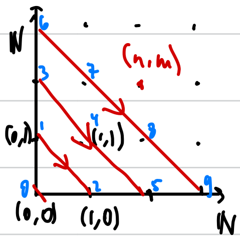
Math 451: Introduction to General Topology
Lecture 3

Examples. (a) $\mathbb{N}^k \equiv \mathbb{N}$.



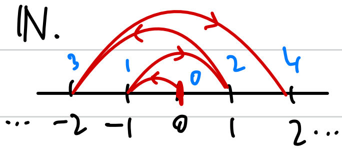
Proof. By induction on k, it is enough to prove that $\mathbb{N}\times\mathbb{N} \equiv \mathbb{N}$. Define $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ by $f(n,m) := \sum_{i=1}^{n+m} i + n = \frac{(n+m)(n+m+1)}{2} + n$. It's not hard to check that this is a bijection by finding the inverse. Left as an exercise. $\square$

(a′) If X is ctbl, then so is X^k for all $k \in \mathbb{N}$.

Proof. X ctbl $\iff \mathbb{N} \twoheadrightarrow X$, so $\mathbb{N}^k \twoheadrightarrow X$. But $\mathbb{N} \equiv \mathbb{N}^k$, so $\mathbb{N} \twoheadrightarrow X^k$, hence X^k is ctbl. $\square$

(b) $\mathbb{Z} \equiv \mathbb{N}$.

Proof.



Let $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(z) := \begin{cases} 2z & \text{if } z \geq 0 \\ 2|z|-1 & \text{if } z < 0 \end{cases}$.
Again, this is easily a bijection.

(c) $\mathbb{Q} \equiv \mathbb{N}$.

Proof. Since $\mathbb{Q} \supseteq \mathbb{N}$, $\mathbb{Q}$ is infinite, so it is enough to show that $\mathbb{N} \twoheadrightarrow \mathbb{Q}$. But $\mathbb{N} \twoheadrightarrow \mathbb{N}^2 \twoheadrightarrow \mathbb{Z}\times\mathbb{N}^+$. We define $f : \mathbb{Z}\times\mathbb{N}^+ \to \mathbb{Q}$ by $(m,n) \mapsto \frac{m}{n}$, so f is a surjection, hence $\mathbb{Z}\times\mathbb{N}^+ \twoheadrightarrow \mathbb{Q}$, thus $\mathbb{N} \twoheadrightarrow \mathbb{Q}$. $\square$

Notation. For a set X, $X^0 := \{\emptyset\}$. Then for $k \geq 1$, put $X^k := X^{k-1} \times X = \underbrace{X \times X \times \ldots \times X}_{k \text{ times}}$.
Thus X^k contains tuples of elements of X of length k.
Denote by $X^{<\mathbb{N}} := \bigcup_{k\in\mathbb{N}} X^k$ and think of this as the set of all finite tuples/words in X.

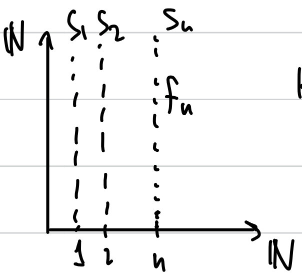
Prop (without AC). If X is ctbl, then so is $X^{<\mathbb{N}}$.

Proof. HW.

The following generalizes the last proposition but requires AC.

Prop (AC). Ctbl union of ctbl sets is ctbl, i.e. if $\mathcal{S}$ is a ctbl set of ctbl sets, then $\bigcup \mathcal{S}$ is ctbl. In particular, for ctbl sets X_n, $n \in \mathbb{N}$, $\bigcup_{n \in \mathbb{N}} X_n$ is ctbl.

Proof. Let $\mathcal{S}$ be ctbl so $\exists$ enumeration $\mathbb{N} \twoheadrightarrow \mathcal{S}$ by $n \mapsto S_n$, i.e. $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, choose an enumeration $f_n : \mathbb{N} \twoheadrightarrow S_n$ and define $f : \mathbb{N} \times \mathbb{N} \to \bigcup \mathcal{S} = \bigcup_{n \in \mathbb{N}} S_n$ by $(n,m) \mapsto f_n(m)$, which is surjective because for any $x \in \bigcup \mathcal{S}$ we have $x \in S_n$ for some $n \in \mathbb{N}$, hence $x = f_n(m)$ for some $m \in \mathbb{N}$. Thus, $\mathbb{N} \equiv \mathbb{N}^2 \twoheadrightarrow \bigcup \mathcal{S}$.



Here we used AC to get a function $n \mapsto f_n$, i.e. to choose a surjection $f_n : \mathbb{N} \twoheadrightarrow S_n$ for each $n \in \mathbb{N}$. □

A real $r \in \mathbb{R}$ is called *algebraic* if it is a root of a polynomial with rational coefficients.

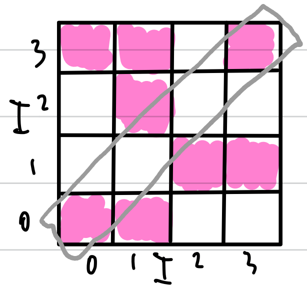
Cor. The set of algebraic numbers is ctbl.

Proof. Each polynomial is determined by a finite tuple of rationals, hence $\mathbb{Q}^{<\mathbb{N}}$ surjects onto the set of polynomials with rational coefficients, so the set of such polynomials is ctbl since $\mathbb{Q}^{<\mathbb{N}}$ is. But each polynomial has only finitely many roots, so the set of all roots (i.e. the set of algebraic number) is a ctbl union of finite sets, hence ctbl. □

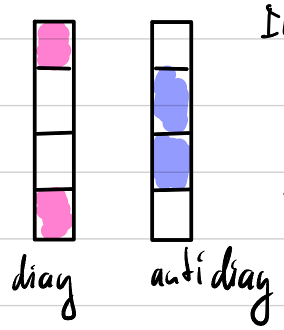
Unctbl sets and Cantor diagonalization.

We would like to prove that are unctbl sets. We will show that $\mathcal{P}(\mathbb{N})$ is unctbl. We will do so via Cantor's *diagonalization method*, which we now describe. Given a matrix M with index set $I \times I$ of entries from $\{0,1\}$, this method produces an

I-indexed vector that doesn't appear in the matrix as a row or a column.



$I = 4 := \{0,1,2,3\}$



Indeed take the antidiagonal:

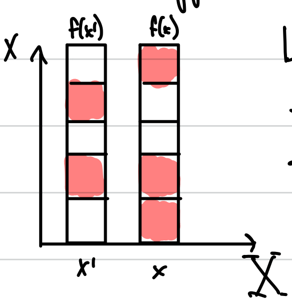
$$\nabla(M)(i) := \begin{cases} 1 & \text{if } M(i,i)=0. \\ 0 & \text{if } M(i,i)=1 \end{cases}$$

Then the vector $\nabla(M)$ is not column nor a row in M.

We use this simple idea to prove:

Theorem (Cantor). For any set X, $X \not\twoheadrightarrow \mathcal{P}(X)$. In particular, $X \not\equiv \mathcal{P}(X)$.

Proof. Suppose toward a contradiction that $\exists\, f : X \twoheadrightarrow \mathcal{P}(X)$.



Let $\nabla_f := \{x \in X : x \notin f(x)\}$ and we show that ∇_f is not in $f(X)$. Indeed for any $x \in X$, we have that $x \in f(x) \iff x \notin \nabla_f$, hence $\nabla_f \neq f(x)$. This contradicts the surjectivity of f. $\square$

Cor. $\mathcal{P}(\mathbb{N})$ is unctbl.

Cor. For any set $X \hookrightarrow \mathcal{P}(X)$ but $\mathcal{P}(X) \not\hookrightarrow X$.

Proof. $X \hookrightarrow \mathcal{P}(X)$ by $x \mapsto \{x\}$ and $\mathcal{P}(X) \hookrightarrow X$ is not true since it would imply $X \twoheadrightarrow \mathcal{P}(X)$, contradicting Cantor's theorem. $\square$

Examples. (a) $\mathcal{P}(X) \equiv 2^X$ for all sets X, hence $X \not\twoheadrightarrow 2^X$. In particular, $2^{\mathbb{N}} \equiv \mathcal{P}(\mathbb{N})$ is unctbl.

(b) $\mathbb{R} \equiv [0,1] \equiv 2^{\mathbb{N}} \equiv \mathcal{P}(\mathbb{N})$ hence $\mathbb{R}$ is unctbl.

Proof. We only need to show $2^{\mathbb{N}} \equiv [0,1]$. It is enough to show that $2^{\mathbb{N}} \hookrightarrow [0,1]$ and $2^{\mathbb{N}} \twoheadrightarrow [0,1]$ since then $[0,1] \hookrightarrow 2^{\mathbb{N}}$ and hence $[0,1] \equiv 2^{\mathbb{N}}$ by the Cantor-Schröder-Bernstein theorem.

$2^{\mathbb{N}} \twoheadrightarrow [0,1]$: Let $f : 2^{\mathbb{N}} \to [0,1]$ be defined by $(x_n)_{n\in\mathbb{N}} \mapsto 0.x_0x_1x_2\ldots$ (binary representation. f is surjective because each real in $[0,1)$ admits a binary representation, including $1 = 0.11111\ldots$ But this isn't injective since $0.01011111111\ldots = 0.0110000\ldots$

$2^{\mathbb{N}} \hookrightarrow [0,1]$: Let $g : 2^{\mathbb{N}} \to \{0,2\}^{\mathbb{N}}$ by $(x_n)_{n\in\mathbb{N}} \mapsto (2x_n)_{n\in\mathbb{N}}$. Map $h: \{0,2\}^{\mathbb{N}} \to [0,1]$ by $(y_n)_{n\in\mathbb{N}} \mapsto 0.y_0 y_1 y_2 \ldots$ treated as a ternary representation. This is an injection since the sequence $y_0, y_1, \ldots$ doesn't have a 1 in it, so there are no ambiguous representations $0.20202222\ldots = 0.20210000\ldots \notin h(\{0,2\}^{\mathbb{N}})$. However h isn't surjective since say $0.11 \notin h(\{0,2\}^{\mathbb{N}})$. In fact the image of h is the standard Cantor set, which we will discuss later. $\square$

Notation. For sets X, Y, instead of writing $X \cong Y$, we write $|X| = |Y|$ and say that X and Y have the same cardinality. We also write $|X| \leq |Y|$ to mean $X \hookrightarrow Y$ and $|X| < |Y|$ to indicate $X \hookrightarrow Y$ but $Y \not\hookrightarrow X$.

In this notation, $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$ but $|\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| = |\mathbb{R}|$. We call this cardinality continuum, a set has cardinality continuum means it is equinumerous with $\mathbb{R}$.

It is natural to wonder if there is a cardinality between $\mathbb{N}$ and $\mathbb{R}$, i.e. if there is a set X such that $|\mathbb{N}| < |X| < |\mathbb{R}|$. The negative answer to this is known as the Continuum Hypothesis (CH). Whether this holds or not turned out to be independent from the set theory axioms math is based on (called ZFC), proved by Gödel and Cohen.